The problem is solved using the method of integral transforms. We omit the intermediate calculations and quote the final result

$$u(r, t) = \frac{3}{2} - \frac{1 - 2\beta(t - r/c)}{2r} A\left(t - \frac{r}{c}\right) + \\ + \frac{c}{2r} \frac{1 + \beta(t - r/c)}{A^2(t - r/c)} \int_0^{t-r/c} A^2(z) \left\{\exp - c \int_z^{t-r/c} \frac{dy}{A(y)}\right\} dz$$

here $z = \tau - a(t)/c$ and $A(z) \equiv a(t(z))$.

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ON THE CONVERGENCE OF THE METHOD OF FINITE ELEMENTS IN THE ANALYSIS OF MEMBRANE DYNAMICS

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The rate of convergence of the method of finite elements in the analysis of natural membrane vibrations is investigated. The analysis is carried out on the basis of elements of two kinds constructed herein.

The convergence criteria of the method of finite elements are formulated in [1, 2]. Their theoretical foundation is given in [3], where it is shown that they are a sufficient condition assuring convergence in energy as the number of elements increases. An analogous proof is presented in [4] for a specific thin plate element.

The rate of convergence of the method is analyzed in [5, 6] in an example of one-dimensional systems. This question is investigated in [7] for a rectangular plate whose two opposite sides are simply supported.

1. To obtain the finite membrane element, let us use the general scheme of the method expounded in [2]. Let the membrane be divided into elements in the shape of parallelograms by line segments (Fig. 1). The points of intersection of the segments are called nodes. Let us examine an individual element with the sides a, b. Let ξ 0 η

denote the oblique coordinate system connected with the element. Let us represent the bending mode of the element as

$$w(\xi, \eta, t) = \psi(\xi, \eta)' C\mathbf{q}(t) \tag{1.1}$$

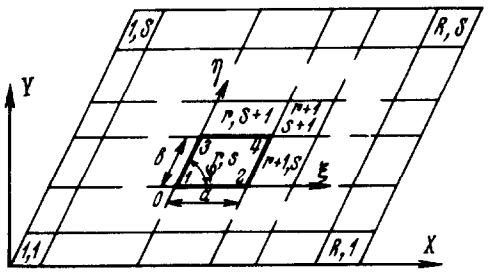

Fig. 1.

where $\mathbf{q}(t)$ is a column vector of the generalized coordinates of the element, $\psi(\xi, \eta)'$ is a line of basis functions, C is a transformation matrix determined from the condition of compatibility of the bending mode and the generalized coordinates.

The vector $\mathbf{Q}(t)$ of the generalized forces corresponding to the coordinates $\mathbf{q}(t)$ is found from the principle of virtual displacements, which is written in the case of free motion as

$$-\delta U(t) + \delta A(t) + \delta \mathbf{q}(t)' \mathbf{Q}(t) = 0 \tag{1.2}$$

where $U(t)$ is the potential elastic strain energy, and $\delta A(t)$ is the work of the inertial forces on the virtual displacement. Taking account of (1.1), the potential energy of an element is expressed as follows:

$$2U(t) = \mathbf{q}(t)' K\mathbf{q}(t)$$

$$K = \frac{T}{\sin\varphi} C' \left[\int_0^a \int_0^b \left(\frac{\partial\psi}{\partial\xi}\frac{\partial\psi'}{\partial\xi} - \cos\varphi \frac{\partial\psi}{\partial\xi}\frac{\partial\psi'}{\partial\eta} - \cos\varphi \frac{\partial\psi}{\partial\eta}\frac{\partial\psi'}{\partial\xi} + \frac{\partial\psi}{\partial\eta}\frac{\partial\psi'}{\partial\eta} \right) d\xi\, d\eta \right] C \tag{1.3}$$

Here K is the stiffness matrix of a finite element, and T is the tension. The virtual work of the inertial forces, taking (1.1) into account, is

$$\delta A = -\delta\mathbf{q}(t)' m \frac{d^2\mathbf{q}(t)}{dt^2}, \qquad m = \sin\varphi C' \int_0^a \int_0^b \rho(\xi, \eta)\psi(\xi, \eta)\psi(\xi, \eta)' d\xi\, d\eta\, C \tag{1.4}$$

where m is the inertial matrix of the element, and $\rho(\xi, \eta)$ is the surface density.

From (1.2) – (1.4) there follows the equation of motion of an individual element subjected to the forces $Q(t)$ with which the elements interact

$$K\mathbf{q}(t) + m\frac{d^2\mathbf{q}}{dt^2} = \mathbf{Q}(t) \tag{1.5}$$

Let us take the displacements in the nodes as the generalized coordinates of the element i.e., we set

$$\mathbf{q}(t)' = \{w^{(j)}(t)\} \quad (j = 1, 2, 3, 4).$$

In this case the components of the vector $Q(t)$ are the concentrated forces acting in the nodes.

We take the system of basis functions in the form

$$\psi(\xi, \eta)' = \left(1, \frac{\xi}{a}, \frac{\eta}{b}, \frac{\xi\eta}{ab}\right) \tag{1.6}$$

Then

$$C = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 1 \end{Vmatrix}. \tag{1.7}$$

Substituting (1.6), (1.7) into (1.3) we find

$$K = \frac{T}{6\sin\varphi}\|k_{lj}\| \qquad (l, j = 1, 2, 3, 4)$$

$$k_{11} = k_{44} = 2\left(\frac{a}{b} + \frac{b}{a}\right) - 3\cos\varphi, \qquad k_{22} = k_{33} = 2\left(\frac{a}{b} + \frac{b}{a}\right) + 3\cos\varphi \tag{1.8}$$

$$k_{12} = k_{21} = k_{34} = k_{43} = \frac{a}{b} - 2\frac{b}{a}, \qquad k_{13} = k_{31} = k_{24} = k_{42} = -2\frac{a}{b} + \frac{b}{a}$$

$$k_{14} = k_{41} = -\left(\frac{a}{b} + \frac{b}{a}\right) + 3\cos\varphi, \qquad k_{23} = k_{32} = -\left(\frac{a}{b} + \frac{b}{a}\right) - 3\cos\varphi$$

Analogously, for $\rho = \text{const}$, from (1.6), (1.7), (1.4) we find

$$m = \frac{\rho ab\sin\varphi}{36}\begin{Vmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{Vmatrix} \tag{1.9}$$

The inertia matrix can be constructed also from the assumption that the mass of the element is concentrated at its nodes. In this case the matrix m is a diagonal matrix.

Using (1.3), (1.4), the stiffness and inertia matrices of finite elements of another shape can be obtained. In particular, elements corresponding to the customary finite-difference partition mesh can be constructed.

2. The following junction conditions hold at the common nodes for adjacent elements (Fig.1):

$$w^{(4)}_{r,s}(t) = w^{(2)}_{r,s+1}(t) = w^{(3)}_{r+1,s}(t) = w^{(1)}_{r+1,s+1}(t) = w^{(r,s)}(t)$$
$$Q^{(4)}_{r,s}(t) + Q^{(2)}_{r,s+1}(t) + Q^{(3)}_{r+1,s}(t) + Q^{(1)}_{r+1,s+1}(t) = 0 \tag{2.1}$$
$$(r = 1, 2, \ldots R;\ s = 1, 2, \ldots S)$$

where r, s are the number of elements and nodes of the model. Appropriate boundary conditions are formulated at the nodes lying on the contour bounding the membrane.

The set of Eqs. (1.5) written for each element, the junction conditions, and the conditions on the boundary form a system describing the free motion of a model of finite elements. The vector of the generalized coordinates of the whole model $\mathbf{q}_\Sigma(t)$, whose components are the displacements of its nodes, can be introduced. Then, on the basis of conditions (2.1), the stiffness K_Σ and inertia m_Σ matrices for the whole model can easily be constructed from the matrices K and m for an individual element. These matrices are square matrices. Their order agrees with the number of free nodes, and therefore with the number of degrees of freedom of the model. If elements with masses concentrated at the nodes are used, then the inertia matrix of the model m_Σ is a diagonal matrix, as for an individual element. We call such a system of elements a model with point masses as contrasted to a model with distributed mass for which the inertia matrix of an element is expressed by (1.9).

The equation of free vibrations of a model of finite elements is

$$K_\Sigma \mathbf{q}_\Sigma(t) + m_\Sigma \frac{d^2\mathbf{q}_\Sigma(t)}{dt^2} = 0 \tag{2.2}$$

3. In some cases, (2.2) is successfully transformed to a convenient form for analytical investigation. Thus, for a model of identical elements with $\varphi = \pi/2$, for which K and m are expressed by (1.8), (1.9), we can write

$$\rho \frac{ab}{6} \frac{d^2}{dt^2} [4\square_{2x}^{(r,s)}w + \square_{xy}^{(r,s)}w + 4\square_{2y}^{(r,s)}w] = \tag{3.1}$$

$$= T\left[\frac{b}{a}(\triangle_{2x}^{(r,s+1)}w + 4\triangle_{2x}^{(r,s)}w + \triangle_{2x}^{(r,s-1)}w) + \frac{a}{b}(\triangle_{2y}^{(r+1,s)}w + 4\triangle_{2y}^{(r,s)}w + \triangle_{2y}^{(r-1,s)}w)\right]$$

$$\square_{2x}^{(r,s)}w = w^{(r+1,s)} + 2w^{(r,s)} + w^{(r-1,s)}$$

$$\square_{xy}^{(r,s)}w = w^{(r+1,s+1)} + w^{(r+1,s-1)} + w^{(r-1,s+1)} + w^{(r-1,s-1)}$$

$$\triangle_{2y}^{(r,s)}w = w^{(r,s+1)} - 2w^{(r,s)} + w^{(r,s-1)}$$

In the case of a rectangular membrane $r = 1, 2 \ldots R - 1$, $s = 1, 2. \ldots S - 1$. The solution of the system (3.1) is

$$w^{(r,s)}(t) = D\exp[i\omega t - i2\alpha r - i2\beta s] \tag{3.2}$$

For a rectangular membrane clamped along the contour, it follows from the boundary conditions that

$$\alpha = \frac{\pi p}{2R}, \qquad \beta = \frac{\pi n}{2S} \qquad (p = 1, 2 \ldots R - 1;\ n = 1, 2 \ldots S - 1) \tag{3.3}$$

where p, n are the number of the fundamental modes.

Substituting (3.2) into (3.1) and taking account of (3.1), we obtain the natural frequencies of a model of $R \times S$ elements with a distributed mass

$$[\omega_{p,n}^{(R\times S)}]^2 = \frac{T}{\rho} 48 \frac{a^{-2}\sin^2\alpha(3 - 2\sin^2\beta) + b^{-2}\sin^2\beta(3 - 2\sin^2\alpha)}{(3 - 2\sin^2\alpha)(3 - 2\sin^2\beta)}$$

If rectangular elements for which the mass is concentrated at the nodes are used, then (3.1) simplifies somewhat and the frequencies of the model become

$$[\omega_{p,n}^{(R\times S)}]^2 = \frac{T}{\rho} \frac{16}{3} [a^{-2}\sin^2\alpha(3 - 2\sin^2\beta) + b^{-2}\sin^2\beta(3 - 2\sin^2\alpha)]$$

As the number of elements of the partition increases, the frequencies of both models tend monotonely to the corresponding frequencies $\omega_{p,n}^{(0)}$ of a continuous system

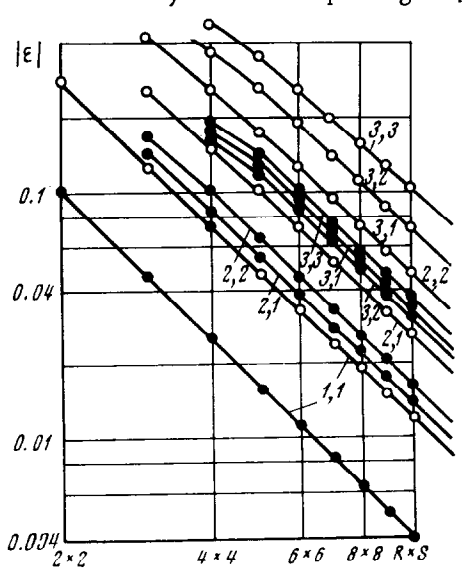

Fig. 2.

$$\lim \omega_{p,n}^{(R\times S)} = \omega_{p,n}^{(0)} = \pi \sqrt{\frac{T}{\rho}\left(\frac{p^2}{A^2} + \frac{n^2}{B^2}\right)}$$

for $R, S \to \infty$

($A = aR$, $B = bS$ are the sides of the membrane).

The relative error of the method of finite elements in computing the natural vibrations of a membrane is defined by the formula

$$\varepsilon_{p,n}^{(R\times S)} = \frac{\omega_{p,n}^{(R\times S)} - \omega_{p,n}^{(0)}}{\omega_{p,n}^{(0)}}$$

Presented in Fig. 2 are values of ε at the first six frequencies of a square membrane. The errors have been calculated for models with different partitions under the condition $R = S$. The number of elements $R \times S$ and the relative error ε are plotted to a logarithmic scale. For a model with distributed mass ε

is noted by dark circles, and for a model with point masses by open circles. For clarification, the values of ε referring to a single tone of the model are connected by segments near which the numbers of the modes are indicated.

From a comparison of the graphs it follows that the model with point masses has considerably greater error. Its frequencies tend to the exact values from below, while for the model with distributed mass from above. The relative error for all modes of the models considered decreases as $(R \times S)^{-1}$ as the number of elements increases. The exception is just the higher modes for a small number of elements.

Table 1.

φ	Mode	Partition				after [9]
		2×2	3×3	4×4	6×6	
75°	1	1.61 1.07*	1.53 1.27*	1.50 1.35*	1.48 1.41*	1.45
	2	— —	2.60 1.68*	2.44 1.89*	2.33 2.06*	2.21
	3	— —	2.72 1.83*	2.59 2.00*	2.50 2.21*	2.38
45°	1	2.20 1.56*	2.04 1.67*	1.99 1.78*	1.94 1.84*	1.88
	2	— —	3.38 2.10*	3.09 2.36*	2.87 2.40*	2.59
	3	— —	3.99 2.55*	3.67 2.81*	3.50 2.90*	3.20

4. The natural frequencies of membranes of arbitrary shape were calculated directly by solving the system (2.2) by using an electronic computer. The frequencies of a circular membrane comprised of finite triangular elements were determined by such a method in [8].

As an illustration here, a membrane in the form of a rhombus for which the solution has been obtained in [9] by the Rayleigh-Ritz method, is examined. In the computation the membrane was divided into rhombic elements of similar shape. The calculations were carried out by a program; the time to determine ten natural modes on a model with 25 degrees of freedom on the M-20 electronic computer did not exceed ten minutes. The first three frequencies of the models are presented in Table 1 for $\varphi = 75°$ and $\varphi = 45°$. The values obtained in [9] are indicated here. The frequencies of the model with point masses (marked with an asterisk in Table 1) lie below the real values, and of the model with concentrated mass, above.

Similar results have also been obtained for membranes of other shapes.

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TORSIONAL VIBRATIONS OF A VISCOELASTIC HALF-SPACE

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A circular rigid stamp is in contact with the surface of a viscoelastic half-space. The stamp performs forced harmonic oscillations around the axis of symmetry. The surface is stress- free everywhere outside the domain of contact.

Approximate expressions are found for the displacement, the stress under the stamp, the moment of the reactive forces acting on the stamp under the assumption of stationarity of the oscillations and their disappearance at infinity.

Sagoci [1] obtained the solution of an analogous problem for an elastic half-space.

1. Let us introduce a cylindrical r, φ, z coordinate system with origin at the center of the contact domain. By analogy with the elastic problem, only the following quantities are not trivial: u_φ, $\tau_{z\varphi}$, $\tau_{r\varphi}$ the angular displacement and the tangential stresses. The motion of the medium is described by the equation for the elastic displacement with two boundary conditions

$$\left[\mu - \int_{-\infty}^{t} K(t-\tau)\,d\tau\right]\left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r^2} + \frac{\partial^2 u_\varphi}{\partial z^2}\right) = \rho\frac{\partial^2 u_\varphi}{\partial t^2}$$

$$u_\varphi = r\,\Phi\exp(i\omega t) \quad \text{for } z = 0,\ r < R$$

$$\tau_{z\varphi} = \left[\mu - \int_{-\infty}^{t} K(t-\tau)\,d\tau\right]\frac{\partial u_\varphi}{\partial z} = 0 \quad \text{for } z = 0,\ r > R$$